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Application of Index Theory in the Study of Special Points of Differential Equations

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Annotation: *This article studies the joint existence of singular points of a system of the form where and* $\frac{dy}{dx}=\frac{Q_n(x,y)}{y+P_n(x,y)}$, and $P_n(x,y)\ Q_n(x,y)$ *are homogeneous polynomials of degree n with constant coefficients . For this purpose the index theory is applied and the characteristics of a singular point are calculated.*

Keywords: *Poincaré index, holomorphic function, one-connected region, piecewise smooth curve, singular point index, zero isoclines.*

INTRODUCTION.

Let us introduce the notion of a singular point index following Poincaré [1] and Bendikson [2]. However, in order to harmonize this method with the well-known topological definition of the index [3], the sign of the Poincaré index will be reversed. Consider the differential equation

$$\frac{dy}{dx}=\frac{Y(x,y)}{X(x,y)}, \qquad (1)$$

where the functions $X(x,y)$ и $Y(x,y)$ are holomorphic in some one-connected region G. Let us take in this domain some closed piecewise smooth curve Γ without self-intersection points and not passing through the singular points of equation (1).

By making a complete single detour along this curve in the counterclockwise direction, we will observe a change in the sign of the right-hand side of equation (1) when the denominator turns to zero, which obviously can take place only at the points of intersection of the contour Γ with the isocline of infinity of this equation.

The number of cases when $\frac{Y(x,y)}{X(x,y)}$, transforming from positive to negative values to infinity, let us denote by K and the number of cases of the transition from negative to positive values of the function to infinity by h. The number

$$j=\frac{K-h}{2} \qquad (2)$$

let us call the index of the closed curve Γ at $K \neq \infty, h \neq \infty$.

Let $M(x_0,\ y_0)$ a singular point of equation (1) in the region G. Let this point be surrounded by some closed piecewise smooth contour contained in G and let this contour contain no other singular points besides the point M. The index of this contour will be called the index of point $M(x, y)$ and denote by j_M. In some cases it is convenient to use the following definition of the singular point index [3]

$j_M = \frac{\Delta(x_0, y_0)}{|\Delta(x_0, y_0)|}$, (3)

where $\Delta(M) = X_x'(x_0, y_0)Y_y'(x_0, y_0) - X_y'(x_0, y_0)Y_x'(x_0, y_0) \neq 0$.

METHODS AND MATERIALS.

In some questions it is convenient to apply index theory to a coarse but easily computable characteristic of a singular point. We apply this theory to study the co-existence of singular points. To do this, we consider the equations

$$\frac{dy}{dx} = \frac{Q_n(x,y)}{y + P_n(x,y)} \qquad (4)$$

where $P_n(x, y)$ и $Q_n(x, y)$-are homogeneous polynomials of power n with constant coefficients a_{ij}, b_{ij}. Introducing the substitution

$$\left. \begin{aligned} x &= \bar{x} + \frac{a_{n_0}}{b_{n_0}} y \\ y &= \bar{y} \end{aligned} \right\} \qquad (5)$$

Let us transform equation (1) to the form

$$\frac{dy}{dx} = \frac{Q_n(x,y)}{y + yP_{n-1}(x,y)}, \qquad (6)$$

where for the new constants and variables the former notations are preserved, and $P_{n-1}(x, y)$ is a homogeneous polynomial $(n-1)$-of degree with coefficients b_{ij}.

Let the equation $\quad Q_n(1, \vartheta) = 0 \qquad (7)$

has n has real solutions, then the isocline of zero will take the form

$$Q_n(x, y) = \sum_{i=1}^{n} (a_i x + b_i y) = 0.$$

In this case, the singular point index can be determined by formula (5) $x = y = 0$.

Lemma. The index of a singular point $x = y = 0$ at n is zero at -even and zero at n-is equal to ± 1.

Proof. If there is an even n numbers k and h will be equal to 1. At odd n, $k = 2$ и $h = 0$ or $k = 0$, $h = 2$. Thus, from here we have $j_{M(0,0)} = 0,\ j_{M(0,0)} = 1,\ j_{M(0,0)} = -1$.

Proof. It follows that the center and focus problem arises only for - odd.

Other singular points of equation (6) will have the form

$$M_i\left(\sqrt[n-1]{\frac{-1}{P_{n-1}(1, v)}},\ v\sqrt[n-1]{\frac{-1}{P_{n-1}(1, v)}}\right)$$

At n -is odd, if equation (4) has n real solutions and for each of them $P_n(1, v_i) < 0$, then equation (4) has $2n + 1$ singular points (including the origin of coordinates), and the coordinates have the form:

$$M_i\left(\sqrt[2m]{-\frac{1}{P_{2m}(1,v_i)}}, \sqrt[2m]{-\frac{v^{2m}}{P_{2m}(1,v_i)}}\right), N_i\left(-\sqrt[2m]{\frac{-1}{P_{2m}(1,v)}}, \sqrt[2m]{-\frac{v_i^{2m}}{P_{2m}(1,v)}}\right)$$

where $i = \overline{1,n}$, $n - 1 = 2m$.

Symmetric singular points will be of the same type. If the origin belongs to the class of singular points K_A, *то из* $2n + 1$ singular points n belongs to k the class K_A, a $n + 1$ class K_C. If the origin belongs to the class $K_C$then $n + 1$ of the singular points will belong to the class K_A, a n class K_C.

Theorem 1. If equation (4) has $2n + 1$ singular points, then $n + 1$ of them belong to the class $K_A$and the other points belong to the class K_C and vice versa.

In the above case the singular points M_i и N_i are only simple, since

$$\Delta(M_i) = \Delta(N_i) = \frac{v_i}{P_{2m}^2(1, v_i)}\left[P_x'(1, v_i)Q_{ny}'(1, v_i) - P_{n-1,y}(1, v_i)Q_n\ (1, v_i)\right] = 0$$

If n −is an even number, a theorem similar to Theorem 1 cannot be formulated. In general, equation (4) in this case can have at most $n + 1$ special points (including the origin). For particular values of n it is possible to establish the joint existence of singular points.

RESULTS AND CONCLUSION.

As an example, let us consider the joint existence of singular points of equation (4) for $n = 3$, at different locations of isoclines of zero and infinity:

$$\frac{dy}{dx} = \frac{b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3}{y(1 + a_{20}x^2 + a_{21}xy + a_{12}y^2)}, \qquad (8)$$

At $b_{n0} > 0$, the origin of coordinates $x = y = 0$ for the equation will be the saddle, and at $b_{n0} < 0$, the focus or center.

If the origin of coordinates is a saddle, then we obtain that in Theorem 1 the class C_k is filled by saddles only, and the class K_A is filled only by antisaddles.

Let $b_{30} > 0$, then by virtue of the above and Theorem 1, if we have seven singular points, then three of them are saddles and the rest are antisaddles.

Equation (8) will take the form

$$b_{30} + b_{21}v + b_{12}v^2 + b_{03}v^3 = 0 \qquad (9)$$

the discriminant of this equation is denoted by D.

$$D = g^2 + p^3$$

where $g = \frac{2b_{12}}{27b_{03}^3} - \frac{b_{12}^3 b_{21}}{6b_{03}^2} + \frac{b_{30}}{2b_{03}}$ (10)

it is easy to see that the number of real solutions of (9) depends on the sign of D.

If $D < 0$ then equation (8) has three valid different solutions, if $D > 0$-one. If $D = 0$-two valid roots, and one of them is twofold.

Consequently, at $D < 0$ equation (8) will take the form

$$\frac{dy}{dx} = \frac{(k_1x-y)(k_2x-y)(k_3x-y)}{y(1+a_{21}x^2+a_{12}xy+a_{03}y^2)} \qquad (11)$$

where k_1, k_2, k_3 –are real roots of equation (11).

Let, in particular, $a_{12} = 0, a_{21} = a_{03} = -1$, then the location of isoclines of zero and infinity can be represented as in Fig. 1.

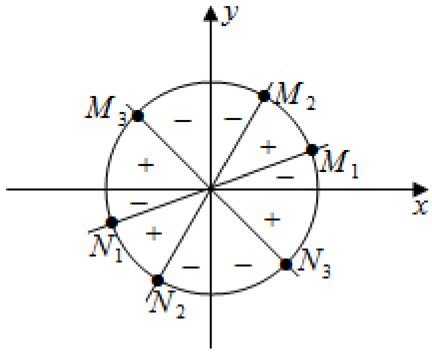


Figure 1.

From where it is clearly seen that the singular points M_1, M_3, N_1, N_3 will be antisets, M_2, N_2 и M (0,0) -saddles. The following theorem is valid here:

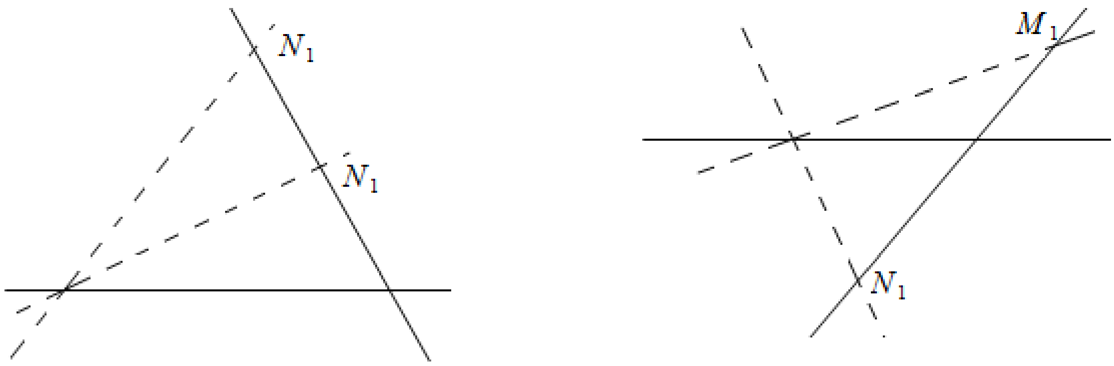
Theorem 2. If equation (8) at $b_{30} > 0$ has seven singular points, then the following cases of their joint existence are possible:

1) Four knots and three saddles;
2) Four tricks and three saddles;
3) Four centers and three saddles;
4) Two knots, two focuses and three saddles;
5) Two knots, two centers and three saddles;
6) Two focuses, two centers and three saddles.

Similar theorems are established by the author when the equation under study has a smaller number of singular points and when the origin is the focus and center. Let us show the above remarks with an example; let $n = 2$. Then equation (4) will take a form

$$\frac{dy}{dx} = \frac{b_{20}x^2 + b_{11}xy + b_{02}y^2}{y(1+a_{20}x+a_{01}y)}$$

This equation has three singular points if the isoclines of zero at infinity are arranged as in Fig. 2. Calculating the indices of the singular points, we have in the case of a) –antisaddle N_1, M_1 or N_1, M_1-saddle, in the case of b) M_1-saddle, N_1-antisaddle, or vice versa. If the singular points of the given equation are simple and it is possible to calculate the coordinates as well, then it is reasonable to calculate the indices of such points using formula (3).



a) b)

Fig. 2

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